Mathematical Meanderings: The Lambert W Function

In algebra, it's likely that you've had to change the subject of the formula, such that a variable appears on its own only on one side of the equation. If there's only one variable involved, then in doing this process you would have 'solved the equation'.

You may wonder if there are any solutions to equations like the following:

$$x = \sin(x)$$

$$x = \ln(x)$$

There's in fact no algebraic solution for these in terms of 'elementary' mathematical functions, i.e. addition, subtraction, multiplication, roots, powers, logs and the trigonometric functions sin, tan and cos. An equation of the form $x = f(x)$ that can't be solved is known as a transcendental equation. $f(x)$ is known as a transcendental function.

Similarly, if there's some expression that again can't be expressed in terms of elementary functions, for example $\int x^x \, dx$, then we say the expression *is not in closed form*. There's a branch of mathematics known as *Galois Theory* that concerns whether expressions are in closed form or not, but examples of expressions that are not in closed form include infinite summations (unless the sum can be simplified, e.g. a convergent geometric series) and most integrals. In addition, think of polynomials, i.e. expressions of the form $a_1 x^{k_1} + a_2 x^{k_2} + \cdots$. We know we can find solutions for any quadratic equation (which may be complex numbers). There's similarly a 'cubic formula' to find solutions for cubics, and even a 'quartic formula'. While solutions for the last of these will have a hideous number of square and cube roots, the solutions are still *exactly* expressible; they are closed form. Interestingly though, Galois Theory can be used to show that there is not necessarily an exact way to express the roots for quintics (i.e. polynomials of order 5) or beyond.

In the Statistics 1 A Level module, you would have looked up z-values in a table to find the probability for example of a randomly chosen person having a height than a certain value, when heights are normally distributed. The reason you require this table is because no calculator would be able to compute this result without approximating it using numerical methods. This is because in finding some area under the Normal curve, we're finding the following integral:

$$P(Z < z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$$

The problem is that e^{-x^2} and its variants cannot be integrated. Interestingly we can find the definite integral $\int_{-\infty}^{+\infty} e^{-x^2} dx$, which remarkably turns out to be $\sqrt{\pi}$ (which is where the $\sqrt{\pi}$ term comes from in the above probability function, to ensure the total area under the curve is 1). However we can't integrate the expression more generally.

If you do Further Maths A Level, you'll learn about Taylor/Macclaurin Expansion in FP2 that allows you to convert any differentiable function to a polynomial expression with an infinite number of terms. Since polynomials can easily be integrated, it allows us to find the integral of the original expression to some required degree of accuracy. Clearly however this value would not be exact given the polynomial is infinitely long.

However, this doesn't stop mathematicians defining functions to represent closed form results. We'll look at two problems:

a) Solve $x = \ln(x)$
b) Find the inverse function of $f(x) = x^x$

Both of these can be solved using the Lambert W function. Imagine we had some function:

$$f(x) = xe^x$$

Then the Lambert W function functions the inverse of this, i.e. if the input is x, then it finds some z such that $x = ze^z$. So for example $W(2)$ gives back a result z such that $2 = ze^z$. Similarly $W(e) = 1$ because $e = 1e^1$.

Thus, the formal definition of $W(x)$ is:

$$x = W(x)e^{W(x)}$$

This is slightly confusing because we've used the function $W(x)$ within the expression rather than on the left side of the equation, but this is because $W(x)$ is an inverse function. This is not closed form, i.e. there's no way we could compute $W(2)$ in some nice way. However, we can get accurate results using numerical methods, and really powerful calculators (such as www.wolframalpha.com) will compute the result for you. Nice!

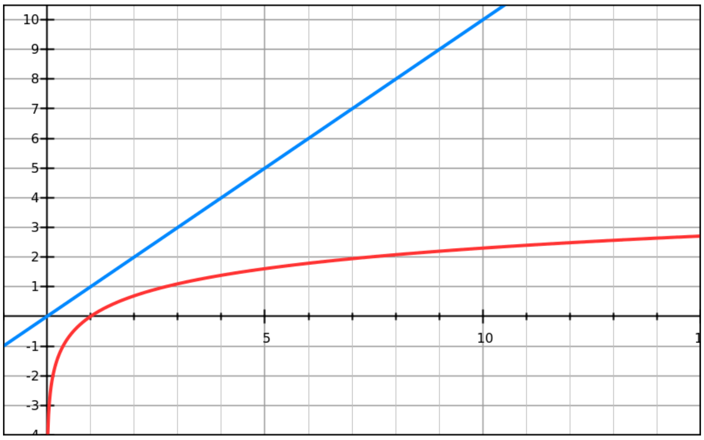
Using the function, we can solve both of our problems:

a) Solve $x = \ln(x)$

The general strategy is to get the equation in the form $a = be^b$, before saying that $W(a) = b$ by definition of the Lambert W function. Getting your expression in this form requires a certain degree of ingenuity. A trick is to exploit laws of logs, notably that $\ln(x) = -\ln\left(\frac{1}{x}\right)$. This allows us to reciprocate the argument of the log where convenient:

$$x = ln(x)$$
$$x = -ln\left(\frac{1}{x}\right)$$
$$-1 = \frac{1}{x} ln\left(\frac{1}{x}\right)$$
$$-1 = ln\left(\frac{1}{x}\right) e^{ln\left(\frac{1}{x}\right)}$$
$$ln\left(\frac{1}{x}\right) = W(-1)$$ (By definition of the Lambert W function.)
$$\frac{1}{x} = e^{W(-1)}$$
$$x = e^{-W(-1)}$$
$$= -W(-1)$$ (Given that $x = \ln x$ and thus $e^x = x$)
$$= 0.318 \pm 1.337i$$ (I just looked up the value for $W(-1)$)

As an aside, note that we would have expected there to be no real solution. You may know from the "Core 1" A Level module, or from elsewhere, that we can solve equalities graphically:



Graphs of $y = x$ and $y = ln(x)$

The graphs never intersect or touch, so there's no real solution. Proving there's no real solution might involve showing that $y = \ln(x)$ never 'catches up' with $y = x$. When x is between 0 and 1, $ln(x)$ is negative while x is positive, so we know they'll never intersect in this region. But when $x > 1$, the gradient of $y = x$ is 1, while the gradient of $\ln(x)$ is $\frac{1}{x} < 1$. And since $\ln(x)$ is only defined in the real domain when $x > 0$, it must therefore never catch up.

b) Find the inverse of $f(x) = x^x$

The inverse of x^2 for example is obvious, i.e. $\sqrt{x}$, as is 2^x, i.e. $\log_2(x)$. The inverse however of x^x is not closed form and requires use of the Lambert W function. This time, it's somewhat easier to get one side of the equation in the form be^b:

$$y = x^x$$
$$\ln(y) = x\, ln(x)$$
$$\ln(y) = \ln(x)\, e^{\ln(x)}$$
$$\ln(x) = W(\ln(y))$$
$$x = e^{W(\ln(y))}$$

Thus $f^{-1}(x) = e^{W(\ln x)}$.